Physics 618 2020

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Last time: We gauged the $SO(2)$ symmetry:

$$\Phi(t) = e^{i\phi(t)} \longrightarrow e^{i\alpha}\, \Phi(t)$$

$$S = \int \left[\frac{1}{2} I \left(\dot{\phi} + A^{(e)} \right)^2 + B \left(\dot{\phi} + A^{(e)}_{(t)} \right) \right] dt$$

$A^{(e)}(t)$: "external" "nondynamical" "background"

$\Phi(t)$: "dynamical" : Integrate over in the path integral.

* Geometrically $d + iA$ = Connection on principal G-bundle over spacetime M = time, 0+1 field theory

$G = \mathbb{R}, SO(2), O(2).$

- $\phi(t) \longrightarrow \phi(t) + \alpha(t)$
- $A^{(e)}(t) \longrightarrow A^{(e)}(t) - \partial_t \alpha(t)$

$A^{(e)} = A^{(e)}(t)\, dt$ $\qquad d + i A^{(e)}$

In general $\quad P \leftarrow G$ = Lie group.

$$\begin{array}{c} P \\ \downarrow \\ M \end{array}$$

3 def's of connection —

* Rule for lifting paths in M to paths in P satisfies gluing — $\|\|$

locally

$$\begin{array}{c} P \\ \downarrow{\scriptstyle \pi} \\ \mathcal{U}_\alpha \subset M \end{array}$$

$$\begin{array}{ccc} P\big|_{\pi^{-1}(\mathcal{U}_\alpha)} & & \mathcal{U}_\alpha \times G \ni (\gamma(t), g(t)) \\ \downarrow & \cong & \downarrow \\ \mathcal{U}_\alpha & & \mathcal{U}_\alpha \ni \gamma(t) \end{array}$$

A connection locally looks like

$$\nabla = d + \underline{A_\alpha} \qquad A_\alpha \in \Omega^1(\mathcal{U}_\alpha ; \underset{\text{Lie}(G)}{\underline{\mathfrak{g}}})$$

$$g^{-1}(t)\left(\frac{d}{dt} + \dot{\gamma}^\mu(t)(A_\alpha)_\mu(\gamma(t))\right) g(t) = 0$$

* Charge conjugation?

$$\phi(t) \to -\phi(t)$$
$$\mathcal{B} \to -\mathcal{B}$$
$$A^{(e)}(t) \to -A^{(e)}(t)$$

Quantum mechanically $\mathcal{B} \to -\mathcal{B}$ can only be a symmetry when $2\mathcal{B} \in \mathbb{Z}$.

* Periodicity in "θ-angle" $\mathcal{B}$

$$\mathcal{B} \to \mathcal{B} + 1$$

Quantum Theory: Value of the action matters

We restore a version of this symmetry by adding a "Chern-Simons term".

$$e^{-S'} = e^{-\int \frac{1}{2} I(\dot\phi + A)^2 - i\mathcal{B}\int(\dot\phi + A) + ik\underbrace{\int A^{(e)}(t)\,dt}_{\text{C-S. term}}}$$

$$(\mathcal{B}, k) \longrightarrow (\mathcal{B}+r,\ k+r) \quad r \in \mathbb{Z}$$

Q.M. $r \in \mathbb{Z}$. for periodicity

Combine with c.c. $\quad A^{(e)} \longrightarrow -A^{(e)}$

$\quad k \longrightarrow -k$

$$(\mathcal{B}, k) \longrightarrow (-\mathcal{B}, -k)$$

$$= (\mathcal{B}+N,\ k+N)$$

Only hope for c.c. when

$$\mathcal{B} = k = N/2 \in \tfrac{1}{2}\mathbb{Z}.$$

Issue: Gauge invariance of Chern-Simons term.

Space of Gauge Fields + Gauge Transformations

In general

gauge group G – compact Lie group.

$\mathcal{G} := \mathrm{Map}(M \to G)$ group of gauge transformations.

($\mathrm{Aut}(P \to M)$, If P is trivializable then $\cong \mathcal{G}$)

$\mathcal{A}$ = space of gauge fields

$\mathcal{A}/\mathcal{G}$ – gauge inequivalent fields.

Case 1: $G = \mathbb{R}$ $\qquad \alpha(t) \in \mathbb{R}$

(Not! $\mathbb{R}/2\pi\mathbb{Z}$)

$\mathcal{G}$: $\mathrm{Map}(M \to \mathbb{R})$

$t \longmapsto \alpha(t)$

Acts on $\mathcal{A}$:

$$A^{(e)}(t) \longrightarrow A^{(e)}(t) - \partial_t \alpha(t)$$

So $M = [t_1, t_2]$ with free b.c.'s

$$\partial_t \alpha(t) = A^{(e)}(t)$$ has a soln

We can always gauge $A^{(e)}(t) = 0$.

$\mathcal{A}/\mathcal{G}$.

<u>Case 2</u> : $G = \mathbb{R}$, $M = \mathbb{R}$

impose $\alpha(t) \longrightarrow 0$ as $t \to \pm\infty$

$$\mathcal{G}' = \{ \text{Map } t \to \alpha(t) \mid \underline{\underline{\alpha(t) \underset{t\to\pm\infty}{\longrightarrow} 0}} \} \subset \mathcal{G}$$

$$\longrightarrow \boxed{\int_{-\infty}^{\infty} A^{(e)}(t)\, dt}$$ is gauge invariant

$[t_1, \cancel{t_2}]$ $\quad \underline{\mathcal{A}/\mathcal{G}'} \cong \underline{\mathbb{R}}$.

Case 3

$$G = SO(2) \cong U(1)$$
$$M = S^1.$$

$$\mathcal{G} = \mathrm{Map}(M \to G)$$
$$= \mathrm{Map}(S^1_{s.t.} \to U(1))$$

as a set of continuous maps we can assign a winding #

$$g(t) : S^1 \longrightarrow S^1$$

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{G}_0 \arrow[r] & \mathcal{G} \arrow[r, "\pi"] & \mathbb{Z} \arrow[r] \arrow[l, bend right, dashed, "s"'] & 1 \end{tikzcd}$$

$\mathcal{G}_0 =$ winding # $= 0$ g.t.'s.

$\pi \circ s = \mathrm{Id}$

s is homom.

$$g(t) = e^{i\alpha(t)} \qquad \alpha(t) : S^1 \to \mathbb{R}$$

single-valued.

$\mathcal{G}_0$ = group of small gauge transformations anything in $\mathcal{G}$ that has nonzero winding number is called a "large gauge transformation"

$$S(\omega) = g_\omega$$

$$g_\omega(t) = \exp\left(2\pi i \omega t/_\beta\right)$$

$$S^1_{s.t.} = [0, \beta] \Big/_{\beta \sim 0}$$

$$g_\omega \cdot g_{\omega'} = g_{\omega+\omega'}$$

$\oint A^{(e)}_{(t)} dt'$ is invt under $\mathcal{G}_0$

is <u>not</u> invt under $\mathcal{G}$!!

$\exp\left(i \oint A^{(e)}_{(t')} dt'\right)$ is invt under $\mathcal{G}$.

Complete gauge invt:

$A^{(e)}(t)$ periodic in $t \sim t+\beta$

$$A^{(e)}(t) = \sum_n e^{2\pi i n t/\beta} A^{(e)}_n$$

$$= A^{(e)}_0 + \tilde{A}^{(e)}$$

We can solve

$$\partial_t \alpha(t) = \tilde{A}^{(e)}(t)$$

for a single-valued $\alpha(t)$

$\therefore$ Using $\underline{\mathcal{G}_0}$ we can always gauge $\underline{A^{(e)}(t) = \mu/\beta}$ constant.

Previous notation $A^{(e)}_t(t)\,\underline{dt}$.

Simplify $A^{(e)}_t(t) \longrightarrow A^{(e)}(t)$

Note:

Under the large gauge trans

$g_w(t)$ $\quad A^{(e)} \to A^{(e)} - g_w^{-1}\partial_t g_w$

shifts $\mu \longrightarrow \mu - 2\pi w \quad w \in \mathbb{Z}$

$$\mathcal{A}/\mathcal{G}_0 \cong \mathbb{R} = \mathcal{A}^{red}$$

$$\mathcal{A}/\mathcal{G} \cong \mathbb{R}/\mathbb{Z} \cong U(1).$$

$\mu \longrightarrow \mu + 2\pi w \quad w \in \mathbb{Z}$

<u>Case 4</u>: $G = O(2)$ gauge c.c.

$O(2) = SO(2) \rtimes \mathbb{Z}_2$ <u>gauge group</u>

$$\mathcal{G}_{O(2)} = \mathcal{G}_{SO(2)} \rtimes \mathbb{Z}_2$$

$$1 \to \mathcal{G}_0 \longrightarrow \mathcal{G}_{O(2)} \longrightarrow \mathbb{Z} \rtimes \mathbb{Z}_2 \cong D_\infty \to$$

action on $\mathcal{A}^{red}$

$\sigma: \mu \to -\mu \qquad s: \mu \longrightarrow \mu + 2\pi$

2 $(\mathcal{A}/\mathcal{G})_{\text{coarse}} = [0, \pi]$

orbifold singularities

$\mu \mapsto -\mu$ around $\mu = 0$

$\mu \to 2\pi - \mu$ around $\mu = \pi$

Better $\mathcal{A}/\mathcal{G}$ = "stack"
"groopoid"

About the Chern-Simons term

$M = S^1_{s.t.}$ $\quad G = U(1)$ $\quad \mathcal{G}$

$$\exp\left(ik \oint_M A^{(e)}(t')\,dt'\right)$$

$\boxed{\text{invt. under } \mathcal{G}_0}$

but for g_w

Not under $\mathcal{G}$:

$$\longrightarrow \exp\left(ik \oint_M A^{(e)}(t')\,dt'\right) \boxed{\exp(2\pi i k w)}$$

$$e^{2\pi i k w} = 1 \quad \text{for all } w \in \mathbb{Z}$$

iff $k \in \mathbb{Z}$.

In fact, for $G = U(1)$ on some spacetimes the action S is <u>NOT</u> gauge invariant!

But this is ok for the quantum theory because in the path integral only $\exp(-S)$ enters

And for quantized level $k \in \mathbb{Z}$

e^{-S} <u>is</u> gauge invariant.

Anomalies: In general in field theory — space $\mathcal{F}$ of "fields"

$$\mathcal{F}^{\text{dynamical}} \longrightarrow \mathcal{F}$$

$$\downarrow \pi$$

$$\mathcal{F}^{\text{backgnd fields}} \qquad = \mathcal{A}/\mathcal{G} \times \{\mathcal{B}\}$$

Often $\mathcal{F} = \mathcal{F}^{bck} \times \mathcal{F}^{dyn.} \times \{k\}$

all "control" parameters can be considered "fields"

Integrate (path integral) over $\mathcal{F}^{dyn.}$

$$Z[g.\phi^{bck}] = \int_{\mathcal{F}^{dyn}} e^{-\overbrace{S[g.\phi^{dyn};g.\phi^{bck}]}^{S[\phi^{dyn};\phi^{bck}]}} \mathrm{Vol}(g.\phi^{dyn})$$

$$\mathrm{Vol}(g.\phi^{dyn}) = \mathrm{Vol}(\phi^{dyn})$$

$Z(\mathcal{B})$ previous section $= Z[\phi^{bck}]$

$Z(\mathcal{B}, A^{(e)}_{(t)})$ in the gauged Q.M.

Suppose $\underline{\underline{\mathcal{G}}}$ acts on $\mathcal{F}$

preserves $S[\phi^{dyn}; \phi^{bck}]$.

Suppose it formally preserves the measure $\mathrm{vol}(\phi^{dyn})$

Then we EXPECT

$$Z[\phi^{bck}] \text{ to be } \mathcal{G}\text{-invt}.$$

$$Z[g\cdot\phi^{bck}] = Z[\phi^{bck}].$$

Path integrals are formal things and need to be defined

e.g. $Z_q \rightsquigarrow \zeta$ - function

It can happen that after defining Z carefully

(regularization, renormalization)

it turns out that well-defined

$\hat{Z}[\phi^{bck}]$ is NOT $\mathcal{G}$-inv.

$\Rightarrow$ "Potential anomaly"

* Sometimes the lack of invce can be removed by physically unimportant redefinitions.

"local counterterms" etc.

* Sometimes the lack of invce CANNOT be removed in this way.

"True anomaly"

$$\exp\left(ik\int A\right) \leftarrow$$

descends to a function on $\mathcal{A}/\mathcal{G}_0$ for any k.

If our gauge group is $G=\mathbb{R}$ this term is not anomalous.

If our gauge group is $G=U(1)$ it does <u>not</u> descend to a function on $\mathcal{A}/\mathcal{G}$ unless $k\in\mathbb{Z}$

if $k\notin\mathbb{Z}$ we say this physical quantity is <u>anomalous</u>.

Look at these ideas in the context of our gauged Q.M.

There won't be any interesting anomalies for $\mathcal{G}_0$.

For simplicity use $\mathcal{G}_0$ to put $A^{(e)}(t) = \mu/\beta$ constant.

Equations of motion are not changed

$$\phi_w(t) = \frac{2\pi w t}{\beta}$$

$$Z(\mu;\beta) =$$

$$e^{ik\mu} \cdot Z_q \cdot \sum_{w\in\mathbb{Z}} e^{-\frac{2\pi^2 I}{\beta}\left(w+\frac{\mu}{2\pi}\right)^2} \cdot e^{-2\pi i \theta\left(w+\frac{\mu}{2\pi}\right)}$$

$$\dot\phi \to \dot\phi + A^{(e)} = \frac{2\pi w}{\beta} + \frac{\mu}{\beta} \qquad w \to w + \frac{\mu}{2\pi}$$

P.S.F. $\Rightarrow$

$$\underline{Z(\mu;B)} = \frac{e^{i(k-B)\mu}}{2\pi} \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2I}(m-B)^2} e^{-i(m-B)\mu}$$

$$= e^{ik\mu}\, \mathrm{Tr}_{\mathcal{H}}\left[e^{-\beta H_B}\, e^{i\mu\cdot Q}\right]$$

$$Q\cdot\Psi_m = m\,\Psi_m$$

If $k\in\mathbb{Z}$ periodic in $\mu\to\mu+2\pi$

No anomaly for $G=SU(2)$

$Z(\mu;B)$ descends to a function on $\mathcal{A}/\mathcal{G}$.

Also $Z(\mu;B) = Z(-\mu;B)$

if $k = B \in \mathbb{Z}$. : No anomaly under $\mathcal{O}(2)$

Gauge Group = O(2)

$$\mathcal{G}/\mathcal{G}_0 \cong D_\infty = \langle \sigma, s \rangle$$

$$\sigma : \mu \longrightarrow -\mu$$
$$s : \mu \longrightarrow \mu + 2\pi$$

$k = \mathcal{B} \in \mathbb{Z}/2$

$k \in \mathbb{Z}$.

$2\mathcal{B}$ even ok.

$2\mathcal{B}$ odd. clash.

$k = \mathcal{B} \in \mathbb{Z} + 1/2$

but we also needed $k \in \mathbb{Z}$

$\Rightarrow\Leftarrow$ Anomaly for O(2).

$$\boxed{\theta = \pi}$$

Actually, by changing the problem again, we can make sense of a $1/2$-integer C. S. term.

$\underline{B = 1/2}$ essential point already clear by looking at leading terms in $\beta \to \infty$ limit

$$Z \longrightarrow \frac{e^{-\beta E_{gnd}}}{2\pi} e^{i(k-\frac{1}{2})\mu} \left(\underline{e^{i\mu/2} + e^{-i\mu/2}} \right)$$

$+ \cdots$

$\underline{\text{If } k=0}$ $\left(1 + e^{-i\mu}\right)$

periodic in $\mu \to \mu + 2\pi$

not invt in $\mu \to -\mu$.

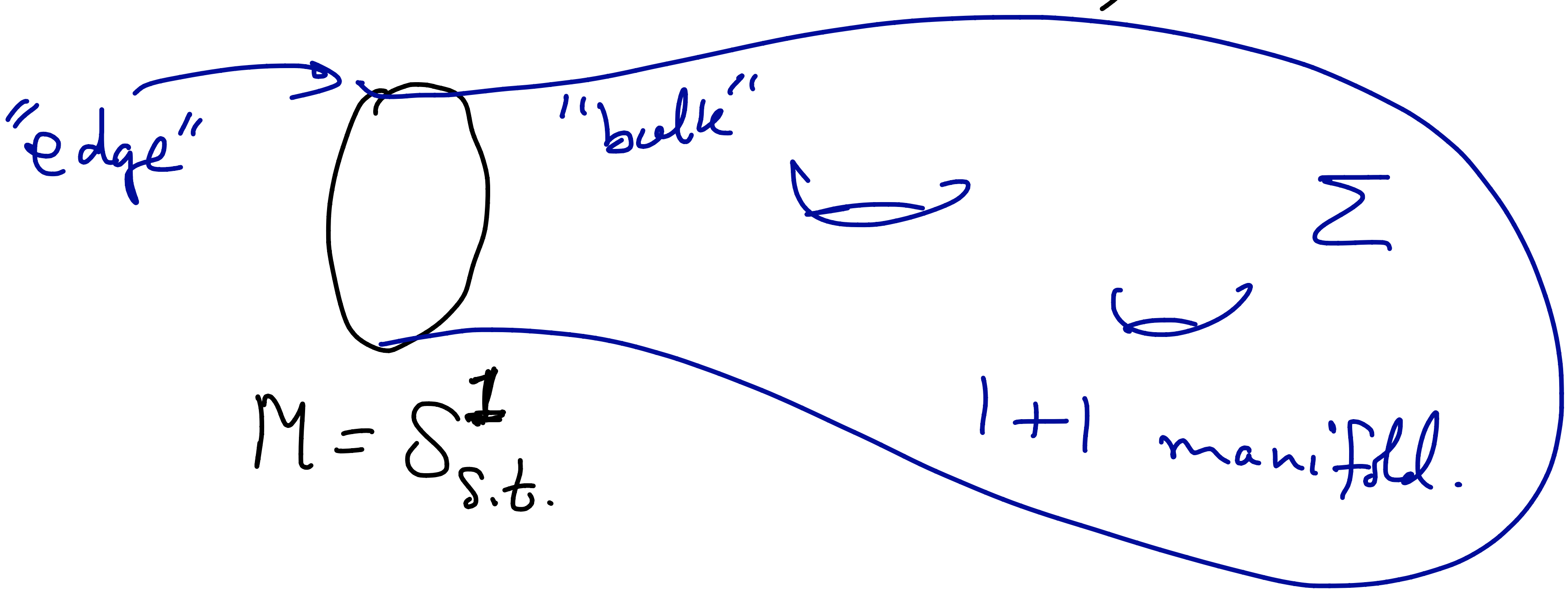
$\underline{\text{If } k=1/2}$: invt $\mu \longrightarrow -\mu$

not invt. under $\mu \to \mu + 2\pi$

So we can have symmetry under s but not σ or under σ but not s depending on choice of k.

In general in the theory of anomalies if there is one definition of Z that is invt. under $\mathcal{H}_1 \subset \mathcal{G}$, and another definition so that Z invt under $\mathcal{H}_2 \subset \mathcal{G}$ but NO def. invt under <u>both</u> $\mathcal{H}_1$ and $\mathcal{H}_2$ we say there is a "<u>mixed anomaly</u>".

Making Sense of $k \notin \mathbb{Z}$ even when $G = U(1)$



$$\exp\left(ik \oint_{S^1} A\right) = \exp\left(ik \int_\Sigma F\right)$$

$F = dA$ is gauge invt

$ik \int_\Sigma F$ makes sense as a gauge invt real number for any k.

(If $\partial\Sigma = \phi$ $\left[F/2\pi\right]$ = image of $c_1(P)$ in H_{dR}.)

Fractional C-S. levels appear

- topological insulators
- fractional quantum Hall effect ("spin Chern-Simons theory")
- susy + sugra + string theory

Heisenberg Extensions

$$1 \to A \to \tilde{G} \to G \to 1$$

Class of central extensions $\quad A = $ Abelian, $G = $ Abelian

but $\tilde{G}$ is — in a sense — "maximally nonabelian"

We've met $\mathrm{Heis}(\mathbb{Z}_n \times \mathbb{Z}_n)$

Q.M. of particle on disc. apprxt to a circle.

Preliminary: Some useful identities for manipulating exponentials of operators.

$$A \in M_n(k)$$

or a suitable operator on $\mathcal{H}$.
(want all powers A^n to exist)

$$\exp(A) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

- $\exp(\alpha A)\exp(\beta A) = \exp((\alpha+\beta) A)$
- $\frac{d}{dt}\left(e^{tA}\right) = A e^{tA} = e^{tA} A$
- $e^{A} e^{B} e^{-A} = e^{(e^{A} B e^{-A})}$

Def: $A \in M_n(k)$

$Ad(A) \in End(M_n(k))$

$Ad(A)$ linear trn on vectorspace of matrices.

$$Ad(A) : B \longrightarrow [A, B]$$

$$(Ad(A))^m : B \to \underbrace{[A, [A, \cdots [A, B] \cdots]]}_{m \text{ times nested.}}$$

Claim: $\boxed{e^A B e^{-A} = \exp(Ad(A))(B)}$

pf: $B(t) = e^{tA} B e^{-tA}$

$B(0) = B$ $\quad$ $B(1) =$ what we want.

$$\frac{d}{dt}\left(\underline{\underline{e^{tA}}}\, B\, \underline{\underline{e^{-tA}}}\right)$$

$$= A \cdot B(t) - B(t) \cdot A$$

$$= Ad(A)\left(B(t)\right)$$

$$\left(\frac{d}{dt}\right)^n B(t)\Big|_{t=0} = Ad(A)^n \left(B(t)\right)\Big|_{t=0}$$

$$= Ad(A)^n (B)$$

$$B(t) = \sum \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n B(t)\Big|_{t=0}$$

$$= \exp\left(t\, Ad(A)\right)(B) \qquad \underline{t=1}.$$

* Suppose now $A(t)$

matrix/operator - valued function of t

In general:

$$\frac{d}{dt} e^{A(t)} \neq \dot{A}(t) e^{A(t)}$$

$$\neq e^{A(t)} \dot{A}(t)$$

Note that, in general, $A(t)$ and $\dot{A}(t)$ don't commute.

Next time: We'll give a useful formula for $\frac{d}{dt} e^{A(t)}$

Also

$$e^{A} e^{B} \neq e^{A+B}$$

when A and B do not commute.

We'll give a formula for $e^{A} e^{B} = e^{C}$

C = function of A and B.

Q.M. of a particle moving on a general Riemannian manifold $(\mathcal{X}, g_{\mu\nu})$

$g_{\mu\nu}(x) = \text{constant}$: free theory.

$$S = \int \frac{1}{2} \overbrace{g_{\mu\nu}(x(t))}^{g_{\mu\nu}(x)=\text{constant}} \dot{x}^{\mu}(t)\, \dot{x}^{\nu}(t)$$

$$\int_{\mathrm{Map}(S^1 \to \mathcal{X})} [dx(t)]\, e^{-S} = Z[g_{\mu\nu}]$$

In general the s.c. appxt. will NOT be exact!

S.C. Appxt. $\quad \delta S = 0$

$\Longleftrightarrow$ closed geodesics

$$\sum_{\substack{\text{closed} \\ \text{geodesics} \\ x_{cl}(t)}} Z_q[x_{cl}(t)]\, e^{-S_{cl}[x_{cl}]}$$

For special case

$$\mathcal{X} = \mathbb{H}/\Gamma$$

$$g_{\mu\nu} = \text{hyperbolic metric.}$$

S.C. appxt. $\Longrightarrow$ Selberg trace formula

Surprising: This is exact!

Related: (not rigorous) Gutzwiller trace formula — generalizes the idea to classically chaotic systems.